Chapter 3

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1) 1_A - measurable $\Longleftrightarrow$ $A \in \mathcal{B}$.

1_A - measurable $\Longleftrightarrow$ $\forall t: \{\omega : 1_A \le t\} \in \mathcal{B}$.

$$\{\omega : 1_A \le t\} = \begin{cases} \Omega, & t \ge 1 \\ A^c, & t < 1 \end{cases} \Longleftrightarrow A^c \in \mathcal{B} \Longleftrightarrow A \in \mathcal{B}$$

4) $\{\omega : X(\omega) \le t\}$ - measurable

$$\{\omega : X(\omega) \le t\} = \bigcup_{x \le t} X^{-1}(x)$$

6) $Y_n = \begin{cases} X, & |X| < n \\ 0, & |X| > n \end{cases}$

$\{Y_n \ne X\} = \{|X| > n\}$

$P\left(\bigcap_{n \ge 1} \{|X| > n\}\right) = 0.$

$P(\{Y_n \ne X\}) \to 0$

$\exists n: \quad P(\{Y_n \ne X\}) < \varepsilon.$

12) $\{x : f(x) \le t\}$ - open or closed ray.

$x_1 \in \Rightarrow x_2 \in$

f-incr.; $x_2 < x_1$

14) $\{x : f(x) < t\}$ - open

$\{x : f(x) > t\} = ?$

18) $P(X \neq Y)$ - small.

$$\{X \in A\} \triangle \{Y \in A\} = \left\{\omega : \begin{matrix} X(\omega) \in A, Y(\omega) \notin A \\ \text{or} \\ X(\omega) \notin A, Y(\omega) \in A \end{matrix}\right\} \subset \{X \neq Y\}$$

$$P(X \in A) = P(\cancel{X \in A, Y \in A}) + P(X \in A, Y \notin A)$$

$$P(Y \in A) = P(\cancel{X \in A, Y \in A}) + P(X \notin A, Y \in A)$$

$$P(\{X \in A, Y \notin A\}) + P(\{X \notin A, Y \in A\}) \leq P(X \neq Y)$$

20) $$\{X_\tau \leq a\} = \bigcup_t \{X_t \leq a, t = \tau\}$$

not countable $$\bigcup \{X_t \leq a\} \cap \{t = \tau\}$$

$\{X_\tau \leq a\}$

$\exists\, t_n \to \tau$ $\lim X_{t_k} \leq a$ - measurable.

rational

$\tau \quad \tau_k : \tau_k \in Q \quad \tau_k = \frac{n}{k}, \ |\tau - \tau_k| < \frac{1}{k}.$

$X_{\tau_k} \to X_\tau.$

22. $$S_n = \sum_{i=1}^{n} X_i$$

$$\tau = \inf\{n > 0 : S_n > 0\}.$$

$$\{\tau = k\} = \left\{ \begin{matrix} S_n < 0, \ n < k \\ S_k > 0 \end{matrix} \right\} = \bigcap_{n=1}^{k-1} \{S_n < 0\} \cap \{S_k > 0\}$$

S_n - random variable. $\quad S_t = \int_0^t X_s \, ds$

$$\{S_\tau < a\} = \bigcup_n \{S_n < a\} \cap \{\tau = n\}$$

25 a) :

$$B_n^{(k)} = \left(\bigcup_{i \geq n} \{ \omega : |X(\omega) - X_i(\omega)| > 2^{-k} \} \right)$$

$$\bigcap_n B_n^{(k)} = \emptyset \Rightarrow \exists n_k : P\left(B^{(k)}_{(n_k)}\right) < \varepsilon 2^{-k}$$

$$P\left(\bigcup_{k=1}^{\infty} B^{(k)}_{n_k} \right) < \varepsilon$$

$$\omega \in \Lambda \setminus \bigcup B^{(k)}_{n_k} \qquad n > n_k \Rightarrow |X(\omega) - X_n(\omega)| \leq 2^{-k}$$